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Crosscap states for orientifolds of euclidean AdS$_3$

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ABSTRACT: Crosscap states for orientifolds of euclidean AdS$_3$ are constructed. We show that our crosscap states describe the same orientifolds which were obtained by the classical analysis. The spectral density of open strings in the system with orientifold can be read from the Möbius strip amplitudes and it is compared to that of the open strings stretched between branes and their mirrors. We also compute the Klein bottle amplitudes.

KEYWORDS: Bosonic Strings, D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

The string theories on AdS_3 have been much investigated in recent years because they can be used to analyze the AdS/CFT correspondence [1] beyond supergravity approximation. These theories have non-trivial NSNS H-flux and can be described by the $\mathrm{SL}(2,\mathbb{R})$ WZW models.[1] The closed string sector of these models has been studied for a decade. The open string sector is now actively investigated by using the classical analysis [2]–[6] and conformal field theory [7]–[13]. The application to the AdS/CFT correspondence was also given in [14].

The unoriented sector of the models can be obtained by including orientifolds. In the terms of conformal field theory, D-branes are described by boundary states [15, 16]. Orientifolds can also be described by crosscap states [17, 18], thus we can analyze the unoriented sector in the similar way as the open string sector. Crosscap states in WZW models were investigated by algebraic way [19, 20, 21] and recently geometric aspects of crosscap states have been studied [22]–[26].

In this paper, we follow the analysis of boundary states in euclidean AdS_3 [7, 10, 12, 13] which use the methods first developed in the context of Liouville field theory with boundary [27]–[33]. The information of boundary states can be given by one point functions on the disk, however they are difficult to calculate directly. If we consider two point functions on the disk, we can obtain some constraints on one point functions by comparing two different expansions [34] and one point functions are determined by solving these

[1]The string theories on euclidean AdS_3, which will be studied in this paper, can be described by the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW models.

constraints. Now we are interested in the case of crosscap states and the same information can be given by one point functions on worldsheet of $\mathbb{RP}^2$. Just as the case of boundary states, we compare two different expansions of two point functions and determine one point functions by solving constraints we obtain [35].

The organization of this paper is as follows. In section 2, we review the closed string sector of string theories on AdS_3 and discuss the geometry of D-branes and orientifolds. In section 3, boundary states are constructed by following the analysis of [12, 13]. In section 4, we obtain constraints of one point functions on $\mathbb{RP}^2$ and find generic solutions. In section 5, we propose crosscap states for orientifolds with the correct geometry. Using the crosscap states, we compute Klein bottle and Möbius strip amplitudes. From the information of the Möbius strip amplitudes, we compare the spectral density of open strings in the system with orientifolds to that of open strings stretched between the mirror branes. The conclusion and discussions are given in section 6. In appendix A, we summarize the useful formulae.

2. Review of string theories on AdS_3

The lorentzian AdS_3 can be given by the hypersurface as

$$(X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = L^2\,, \tag{2.1}$$

where L is the radius of AdS_3 space and we will set $L = 1$ for a while. The euclidean AdS_3 (H_+^3) can be obtained by the Wick rotation $X^3 = iX_E^3$. This space can be also realized by $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ group manifolds as

$$g = \begin{pmatrix} \gamma\bar\gamma e^{\phi} + e^{-\phi} & -\gamma e^{\phi} \\ -\bar\gamma e^{\phi} & e^{\phi} \end{pmatrix} \tag{2.2}$$

whose metric can be given by

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar\gamma\,. \tag{2.3}$$

The coordinate ϕ describes the radial direction and $\phi \to \infty$ corresponds to the boundary of euclidean AdS_3, where γ and $\bar\gamma$ become the coordinates of the boundary.

The string theory on this background is given by the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW models and they were well investigated.[2] The important class of functions on H_+^3 is given by

$$\Phi_j(x,\bar x, z, \bar z) = \frac{1-2j}{\pi}\left(\frac{1}{e^{-\phi} + |\gamma - x|^2 e^{\phi}}\right)^{2j}, \tag{2.4}$$

which have the spin j under the transformation of $\mathrm{SL}(2,\mathbb{C})$ and $x, \bar x$ labels some quantum numbers.[3] The $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZW models have conserved currents and primary states. The primary states correspond to the functions (2.4) and transform as

$$J^a(z)\Phi_j(x,\bar x, w, \bar w) \sim -\frac{\mathcal{D}^a}{z-w}\Phi_j(x,\bar x, w, \bar w)\,. \tag{2.5}$$

[2] See, for example, [37].

[3] The labels x, $\bar x$ can be also identified as the boundary coordinates in the sense of AdS/CFT correspondence [1].

Here $a = \pm, 3$ and $\mathcal{D}^a$ are given by

$$\mathcal{D}^+ = \frac{\partial}{\partial x}\,, \qquad \mathcal{D}^3 = x\frac{\partial}{\partial x} + j\,, \qquad \mathcal{D}^- = x^2\frac{\partial}{\partial x} + 2jx\,. \tag{2.6}$$

The anti-holomorphic currents are defined in the same way. The energy momentum tensor is given by Sugawara construction as

$$T = \frac{1}{2(k-2)}\left(J^+J^- + J^-J^+ - 2J^3J^3\right), \tag{2.7}$$

where k is the level of the models which is related to the radius L. The conformal weights of the primary fields (2.4) can be calculated by this energy momentum tensor as

$$\Delta_j = -\frac{j(j-1)}{k-2}\,. \tag{2.8}$$

The normalizable mode has the spin $j = 1/2 + is$, $s \in \mathbb{R}$ and the Hilbert space of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW models can be decomposed by this label. Precisely speaking, this is the double counting because the states with j and $1-j$ are related as

$$\Phi_j(x,\bar{x},z,\bar{z}) = R(j)\frac{2j-1}{\pi}\int d^2y|x-y|^{-4j}\Phi_{1-j}(y,\bar{y},z,\bar{z})\,, \tag{2.9}$$

where

$$R(j) = \nu^{1-2j}\frac{\Gamma(1-u(2j-1))}{\Gamma(1+u(2j-1))}\,, \qquad \nu = \frac{\Gamma(1-u)}{\Gamma(1+u)}\,, \tag{2.10}$$

and we use $u = 1/(k-2)$. Two point functions were calculated in [38] as

$$\begin{aligned}\langle\Phi_j(x,\bar{x},z,\bar{z})\Phi_{j'}(y,\bar{y},w,\bar{w})\rangle &= \\ &= \frac{1}{|z-w|^{4\Delta_j}}\delta^{(2)}(x-y)\delta\left(j+j'-1\right) + \frac{B(j)}{|z-w|^{4\Delta_j}|x-y|^{4j}}\delta\left(j-j'\right),\end{aligned} \tag{2.11}$$

where

$$B(j) = \nu^{1-2j}\frac{2j-1}{\pi}\frac{\Gamma(1-u(2j-1))}{\Gamma(1+u(2j-1))}\,. \tag{2.12}$$

In order to include branes and orientifolds, it is convenient to introduce other parametrizations of lorentzian AdS_3 space as

$$X_1 = \cosh\psi\sinh\omega\,, \qquad X_2 = \sinh\psi\,, \qquad X_0 + iX_3 = \cosh\psi\cosh\omega e^{it}\,, \tag{2.13}$$

where the metric is given by

$$ds^2 = d\psi^2 + \cosh^2\psi\left(-\cosh^2\omega dt^2 + d\omega^2\right)\,. \tag{2.14}$$

The euclidean version of AdS_3 space can be obtained by the Wick rotation $t_E = it$ just as before.

Maximally symmetric branes were investigated in [2, 3, 4] classically and the geometry of physical branes was identified as AdS_2 space, which corresponds to constant ψ slice in the coordinates (2.13). The open string stretched between two branes can be described by worldsheet with boundary, and hence we have to assign boundary conditions to currents. The boundary conditions for maximally symmetric branes are given in the terms of boundary states as[4]

$$(J^a_n + \bar{J}^a_{-n})|B\rangle = 0\,. \tag{2.15}$$

In the next section, we will construct this type of boundary states by following the analysis of [12, 13].

The geometry of the orientifolds was already discussed in [24]. Orientifold operations are given by the combination of worldsheet parity reversal ($\Omega : \sigma \to 2\pi - \sigma$) and space time $\mathbb{Z}_2$ isometries ($h = \mathrm{diag}(\pm1, \pm1, \pm1, \pm1)$ in the coordinates (2.1)). However, in order to preserve the non-trivial H-flux, we have to choose h which reverse the orientation of manifolds. Moreover orientifolds must be time-like surfaces, therefore we can only use $h = (+1, +1, -1, +1)$.[5] This means $\psi = 0$ slice in the coordinates (2.13), thus the geometry of orientifolds is AdS_2 space. The corresponding crosscap states obey the conditions like boundary states as

$$\left(J^a_n + (-1)^n \bar{J}^a_{-n}\right)|C\rangle = 0\,. \tag{2.16}$$

In section 4 and 5, we will construct this type of crosscap states and study their properties.

3. Boundary states for AdS_2 branes

Boundary states can be constructed from the information of one point functions on the disk with some boundary conditions. The ansatz for one point function obeying the condition (2.15) was proposed in [12, 13] as

$$\langle \Phi_j(x, \bar{x}, z, \bar{z})\rangle_\Theta = \frac{U^{\pm}_\Theta(j)}{|x-\bar{x}|^{2j}|z-\bar{z}|^{2\Delta_j}}\,, \tag{3.1}$$

where $+$ for $x_2 > 0$ and $-$ for $x_2 < 0$ ($x = x_1 + ix_2$). We have used Θ as the label of boundary conditions. The solution which obeys the boundary conditions (2.15) is locally given by $|x-\bar{x}|^{-2j}$. This solution has singular points along $\mathrm{Im}\,x = 0$, thus we can use the different ansatz across this line. From the viewpoint of AdS/CFT correspondence, the AdS_2 branes can be domain walls to the boundary CFT at $\mathrm{Im}\,x = 0$, therefore the discontinuity can be allowed. However the coefficients U^+_Θ and U^-_Θ are not independent but related by the reflection relations (2.9) as ($y = y_1 + iy_2$)

$$\frac{U^{\pm}_\Theta(j)}{|x-\bar{x}|^{2j}} = R(j)\frac{2j-1}{\pi}\left(\int_{y_2>0} d^2y \frac{U^+_\Theta(1-j)|x-y|^{-4j}}{|y-\bar{y}|^{2(1-j)}} + \int_{y_2<0} d^2y \frac{U^-_\Theta(1-j)|x-y|^{-4j}}{|y-\bar{y}|^{2(1-j)}}\right). \tag{3.2}$$

Integrating this equations, we obtain the following simple relations as

$$U^{\pm}_\Theta(j) = R(j)U^{\mp}_\Theta(1-j)\,. \tag{3.3}$$

[4]The notation of currents is different from that of [8], so the same boundary conditions are given in the different way.

[5]We can use the ones rotated by the symmetries.

Rewriting the coefficients as

$$U_{\Theta}^{\pm}(j) = \Gamma\left(1 - u(2j-1)\right) \nu^{1/2-j} f_{\Theta}^{\pm}(j)\,, \tag{3.4}$$

we get the relations for $f_{\Theta}^{\pm}(j)$ as

$$f_{\Theta}^{\pm}(j) = f_{\Theta}^{\mp}(1-j)\,. \tag{3.5}$$

As we mentioned in the introduction, one point functions are difficult to calculate and hence we utilize two point functions. However general two point functions are also difficult to calculate, therefore we make use of the state $\Phi_{-1/2}$ belonging to the degenerate representation. This state has the properties which make the analysis very simple as

$$\partial_x^2 \Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z}) = 0\,, \tag{3.6}$$

and hence the operator product expansions with Φ_j include only two terms

$$\begin{aligned}\Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z})\Phi_j(y,\bar{y},w,\bar{w}) \sim{}& C_+(j)|z-w|^{2u(1-j)}|x-y|^2\Phi_{j+\frac{1}{2}}(y,\bar{y},w,\bar{w}) + \\ &+ C_-(j)|z-w|^{2uj}\Phi_{j-\frac{1}{2}}(y,\bar{y},w,\bar{w})\,.\end{aligned} \tag{3.7}$$

The coefficients were obtained in [38] as

$$\begin{aligned} C_+(j) &= \nu\frac{\Gamma(-u)\Gamma(1+2u)}{\Gamma(-2u)\Gamma(1+u)}\,, \\ C_-(j) &= \frac{\Gamma(-u)\Gamma(1+2u)\Gamma(u(2j-2))\Gamma(1-u(2j-1))}{\Gamma(-2u)\Gamma(1+u)\Gamma(u(2j-1))\Gamma(1-u(2j-2))}\,. \end{aligned} \tag{3.8}$$

From these reasons we can calculate the following two point functions

$$\left\langle \Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z})\Phi_j(y,\bar{y},w,\bar{w}) \right\rangle_{\Theta}\,, \tag{3.9}$$

which will be used to obtain the constraints on one point functions. If one state approaches to another state ($z \to w$, $x \to y$), it is natural to use the previous OPE (3.7) and if the states become close to the boundary, it is natural to expand by boundary operators. Comparing the two expansions, we find the constrains as

$$2\sinh\Theta \cdot f_{\Theta}^{+}(j) = f_{\Theta}^{+}\left(j+\frac{1}{2}\right) - f_{\Theta}^{+}\left(j-\frac{1}{2}\right), \tag{3.10}$$

and general solutions can be given by the linear combinations of

$$f_{\Theta}^{\pm}(j) = e^{\pm(2\pi i n+\Theta)(2j-1)}\,, \qquad e^{\pm(\pi i(2n+1)-\Theta)(2j-1)}\,, \qquad n \in \mathbb{Z}\,. \tag{3.11}$$

The authors [12, 13] proposed the solutions which correspond to the boundary states for AdS_2 branes as

$$f_{\Theta}^{\pm}(j) = Ce^{\pm\Theta(2j-1)}\,, \tag{3.12}$$

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where C is some constant[6] independent of j. The corresponding boundary states are given by

$$|\Theta\rangle_C = \int_{\frac{1}{2}+i\mathbb{R}_+} dj \left(\int_{x_2>0} d^2x \frac{U^+_\Theta(1-j)}{|x-\bar{x}|^{2(1-j)}} |j,x,\bar{x}\rangle_I + \int_{x_2<0} d^2x \frac{U^-_\Theta(1-j)}{|x-\bar{x}|^{2(1-j)}} |j,x,\bar{x}\rangle_I \right), \tag{3.13}$$

where $|j,x,\bar{x}\rangle_I$ are "Ishibashi" boundary states based on the primary states $|j,x,\bar{x}\rangle$.

In fact, the terminology of "Ishibashi" boundary states is not accurate. The usual Ishibashi boundary states are defined by

$$|j\rangle_I = \sum_{I,J} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |\Phi_j\rangle , \tag{3.14}$$

which obey the conditions (2.15) [15, 9]. The labels I, J are defined by the ordered set of (a_i, n_i) ($n_i \geq 0$) and $J_I = J^{a_1}_{n_1}, \ldots, J^{a_r}_{n_r}$. The coefficients M_{IJ} are defined by $M_{IJ} = \langle \Phi_j | J_I J_{-J} | \Phi_j \rangle$. In our case, there is a discontinuity along $\operatorname{Im} x = 0$, thus the decomposition by the label $x, \bar{x}$ might be needed. Then the "Ishibashi" boundary states are defined by using the basis $|j,x,\bar{x}\rangle$ and restricting the summation to non-zero modes as

$$|j,x,\bar{x}\rangle_I = \widetilde{\sum_{I,J}} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |j,x,\bar{x}\rangle . \tag{3.15}$$

The geometry of the branes which the boundary states describe can be seen in the large k limit by scattering with the closed string states $|g\rangle$ which are localized at g [36] as

$$\langle g | j,x,\bar{x}\rangle = \Phi_j(x,\bar{x}|g) . \tag{3.16}$$

The overlaps with the boundary states were calculated in [12, 13] and the results are given by

$$\lim_{k\to\infty} \langle g | \Theta \rangle_C = \frac{1}{4\pi} \delta(\sinh\psi - \sinh\Theta) , \tag{3.17}$$

thus we can see that the boundary states (3.13) describe AdS_2 branes at $\psi = \Theta$.

The annulus amplitudes can be given by the overlaps between two boundary states as

$$\begin{aligned} {}_C\langle\Theta_1|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\Theta_2\rangle_C = & \int_{\frac{1}{2}+i\mathbb{R}_+} dj \left(\int_{x_2>0} d^2x \frac{U^+_{\Theta_1}(j) U^+_{\Theta_2}(1-j)}{|x-\bar{x}|^2} + \right. \\ & \left. + \int_{x_2<0} d^2x \frac{U^-_{\Theta_1}(j) U^-_{\Theta_2}(1-j)}{|x-\bar{x}|^2} \right) \frac{\tilde{q}^{u(j-\frac{1}{2})^2}}{\eta(\tilde{q})^3} , \end{aligned} \tag{3.18}$$

where $\tilde{q} = e^{2\pi i(-1/\tau)}$ is the closed string modulus and c is the central charge of the model. Using the modular transformation, the amplitudes can be transformed into the open string channel. The modular transformation is given by

$$\frac{s\tilde{q}^{us^2}}{\eta(\tilde{q})^3} = 2\sqrt{2u} \int_0^\infty ds' \sin(4\pi u s s') \frac{s' q^{us'^2}}{\eta(q)^3} , \qquad q = e^{2\pi i \tau} , \tag{3.19}$$

[6] We will set $C = 1$ because it does not affect the arguments below.

where we use the modular transformation of η function (A.10) and the annulus amplitudes can be rewritten as

$$
\begin{aligned}
&{}_C\langle\Theta_1|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\Theta_2\rangle_C = \\
&\quad = \int d^2x \frac{1}{|x-\bar{x}|^2} \int_0^\infty ds' \frac{\sqrt{2u}\pi \sinh(2\pi s')s'}{\cosh\left(\frac{1}{u}(\Theta_1-\Theta_2)\right) + \cosh(2\pi s')} \frac{q^{us'^2}}{\eta(q)^3}\,. \qquad (3.20)
\end{aligned}
$$

The x integral would be divergent, but this can be interpreted as the divergence due to the integration of the infinite worldvolume of AdS_2 branes. The spectral density can be read from the annulus amplitude as

$$
\rho(s) \propto \frac{\sinh(2\pi s)s}{\cosh(\frac{1}{u}(\Theta_1-\Theta_2)) + \cosh(2\pi s)}\,, \qquad (3.21)
$$

where the spectrum belongs to the continuous series. We should note that the coefficients are not integers but non-negative real numbers, contrary to the rational conformal field theory case.

4. Constraints for one point functions on $\mathbb{RP}^2$

Orientifolds can be described by crosscap states, which can be obtained by the information of one point functions on $\mathbb{RP}^2$. The ansatz for one point function can be given just like the case of the boundary states as

$$
\langle\Phi_j(x,\bar{x},z,\bar{z})\rangle_{\mathbb{RP}^2} = \frac{U_C^{\pm}(j)}{|x-\bar{x}|^{2j}|1+z\bar{z}|^{2\Delta_j}}\,. \qquad (4.1)
$$

The x dependence can be determined by the conditions (2.16) and the ansatz of $+$ and $-$ are given for $x_2 > 0$ and $x_2 < 0$, respectively. The discontinuity across $\mathrm{Im}\, x = 0$ exists for the same reasons as the boundary states case. The z dependence can be obtained by the mirror technique of unoriented worldsheet. When we construct the boundary states for D-branes, we use the disk amplitude, which is essentially identical to the amplitude on the upper half plane. By using the mirror image technique, we can map the upper half plane to the whole plane and vice versa. The reflection $I(z) = \bar{z}$ is used and it gives the fixed line at $\mathrm{Im}\, z = 0$. Now we are considering the unoriented worldsheet. In this case, the reflection $I(z) = -1/\bar{z}$ is used and the worldsheet can be restricted to the upper half plane. This action gives no fixed line and the geometry becomes $\mathbb{RP}^2$. The one point function on the upper half plane can be given by the two point function on the whole plane by making use of this reflection.

The analysis used in the previous section can be applied to the case of orientifolds [35]. We again utilize the two point function on $\mathbb{RP}^2$ with the primary $\Phi_{-1/2}$ as

$$
\left\langle \Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z})\Phi_j(y,\bar{y},w,\bar{w}) \right\rangle_{\mathbb{RP}^2}\,. \qquad (4.2)
$$

Just as the case of the boundary states, we expand this quantity by two different ways and we obtain the constraints by comparing the two expansions. One way to express this quantity is to make use of the operator product expansion (3.7). This expansion is natural when the two primaries are close ($z \to w$, $x \to y$) and it is given by[7]

$$\begin{aligned}\left\langle \Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z})\Phi_j(y,\bar{y},w,\bar{w})\right\rangle_{\mathbb{RP}^2} &= \frac{|y-\bar{y}|^{-1-2j}}{|x-\bar{y}|^{-2}}\frac{|1+w\bar{w}|^{-\frac{3u}{2}-2\Delta_j}}{|1+z\bar{w}|^{-3u}}\times \\ &\times\left(C_+(j)U_C^+\left(j+\frac{1}{2}\right)F_+(\chi,\eta)+C_-(j)U_C^+\left(j-\frac{1}{2}\right)F_-(\chi,\eta)\right),\end{aligned} \tag{4.3}$$

where we have defined the cross ratios as

$$\chi = \frac{|x-y|^2}{(x-\bar{x})(y-\bar{y})}\,, \qquad \eta = \frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})}\,. \tag{4.4}$$

The functions $F_\pm(\chi,\eta)$ are four point conformal blocks which were calculated in [38] as

$$\begin{aligned}F_+(\chi,\eta) &= \eta^{u(1-j)}(1-\eta)^{uj}\Bigg(\chi F(-u,u+1,1-u(2j-1);\eta)+ \\ &\qquad + \frac{uz}{1-u(2j-1)}F(1-u,u+1,2-u(2j-1);\eta)\Bigg)\,, \\ F_-(\chi,\eta) &= \eta^{uj}(1-\eta)^{uj}\Bigg(\frac{2j\chi}{1-2j}F(1+2uj,2u(j-1),1+u(2j-1);\eta)+ \\ &\qquad + F(2uj,2u(j-1),u(2j-1);\eta)\Bigg)\,.\end{aligned} \tag{4.5}$$

We use $F(a,b,c;\eta)$ as the hypergeometric functions whose properties are summarized in appendix A.

The other way to express the two point function (4.2) is to use the operator product expansion between $\Phi_{-1/2}$ and the mirror image of Φ_j. It is natural to use this expansion when z approaches to $-1/\bar{w}$ and x approaches to $\bar{y}$, namely, when it can be expanded by $1-\eta$ and $1-\chi$. Noticing $\mathrm{Im}\,\bar{y}<0$ from the assumption, we obtain

$$\begin{aligned}\left\langle \Phi_{-\frac{1}{2}}(x,\bar{x},z,\bar{z})\Phi_j(y,\bar{y},w,\bar{w})\right\rangle_{\mathbb{RP}^2} &= \frac{|y-\bar{y}|^{-1-2j}}{|x-\bar{y}|^{-2}}\frac{|1+w\bar{w}|^{-\frac{3u}{2}-2\Delta_j}}{|1+z\bar{w}|^{-3u}}\times \\ \times\left(C_+(j)U_C^-\left(j+\frac{1}{2}\right)\right.&\left.F_+(1-\chi,1-\eta)+C_-(j)U_C^-\left(j-\frac{1}{2}\right)F_-(1-\chi,1-\eta)\right).\end{aligned} \tag{4.6}$$

Comparing two expansions (4.3) and (4.6), we get the constraints of the coefficients $U_\Theta^\pm(j)$. When comparing them, we use the following relations obtained by using the formula for hypergeometric functions in appendix A as

$$\begin{aligned}F_+(\chi,\eta) &= \frac{\Gamma(1-u(2j-1))\Gamma(1-u(2j-1))}{\Gamma(1-u(2j-2))\Gamma(1-2uj)}F_-(1-\chi,1-\eta)- \\ &\quad -\frac{\Gamma(1-u(2j-1))\Gamma(u(2j-1))}{\Gamma(-u)\Gamma(1+u)}F_+(1-\chi,1-\eta)\,,\end{aligned}$$

[7] We assume $\mathrm{Im}\,y>0$ and hence we use $U^+(j)$. If we use $\mathrm{Im}\,y<0$, we should replace $\pm$ by $\mp$ in the following discussions.

$$F_-(\chi,\eta) = \frac{\Gamma(u(2j-1))\Gamma(u(2j-1))}{\Gamma(u(2j-2))\Gamma(2uj)} F_+(1-\chi,1-\eta) - \\ - \frac{\Gamma(1+u(2j-1))\Gamma(-u(2j-1))}{\Gamma(-u)\Gamma(1+u)} F_-(1-\chi,1-\eta)\,. \tag{4.7}$$

Using the expressions of $C_\pm(j)$ (3.8) and $f^\pm_C(j)$ defined by

$$U^\pm_C(j) = \Gamma(1-u(2j-1))\nu^{1/2-j} f^\pm_C(j)\,, \tag{4.8}$$

we obtain the following constraints as

$$f^+_C\left(j-\frac{1}{2}\right)\sin(2\pi uj) + f^+_C\left(j+\frac{1}{2}\right)\sin(\pi u) = f^-_C\left(j+\frac{1}{2}\right)\sin(\pi u(2j-1))\,, \\ f^+_C\left(j+\frac{1}{2}\right)\sin(2\pi u(j-1)) - f^+_C\left(j-\frac{1}{2}\right)\sin(\pi u) = f^-_C\left(j-\frac{1}{2}\right)\sin(\pi u(2j-1))\,. \tag{4.9}$$

General solutions of these equations are given by

$$f^\pm_C(j) = \pm C(j)\cos\left(\pi u\left(j-\frac{1}{2}\right)\right)\,, \tag{4.10}$$

where $C(j)$ are sort of phase factors which satisfy

$$C(j+1) = -C(j)\,, \qquad C(1-j) = -C(j)\,. \tag{4.11}$$

5. Crosscap states for orientifolds

The AdS_2 orientifolds are located on $\psi = 0$ in the coordinates (2.13) and hence we have to construct the crosscap states which reproduce the geometry in the classical limit ($k \to \infty$). Therefore we propose the following solutions

$$U^+_C\left(j=\frac{1}{2}+is\right) = \nu^{-is}\cosh(\pi us)\Gamma(1-2ius)\,, \tag{5.1}$$

where we restrict the label j to the normalizable mode. Because of this restriction we can use trivial phase factors which still satisfy (4.11) along the shift of the real part of j. At this stage, we can say at most that the solutions should be (5.1) in the classical limit. However we will see below that the spectral density of open strings in the system with orientifold reproduces that of open strings stretched between the mirror branes. From these reasons we believe that our solutions are correct ones. The crosscap states are constructed by these solutions as

$$|C\rangle_C = \int_{\frac{1}{2}+i\mathbb{R}_+} dj \left(\int_{x_2>0} d^2x \frac{U^+_C(1-j)}{|x-\bar{x}|^{2(1-j)}}|C;j,x,\bar{x}\rangle_I + \right. \\ \left. + \int_{x_2<0} d^2x \frac{U^-_C(1-j)}{|x-\bar{x}|^{2(1-j)}}|C;j,x,\bar{x}\rangle_I\right)\,, \tag{5.2}$$

where $|C;j,x,\bar{x}\rangle_I$ are "Ishibashi" crosscap states based on the primary states $|j,x,\bar{x}\rangle$. These states are defined just as the "Ishibashi" boundary states.

The spectrum of closed strings in the system with orientifold can be read from the Klein bottle amplitude. This amplitude can be obtained from the overlap between two crosscap states and it is given by

$$\begin{aligned}
{}_C\langle C|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|C\rangle_C &= \int_{\frac{1}{2}+i\mathbb{R}_+} dj \left(\int_{x_2>0} d^2x \frac{U_C^+(j)U_C^+(1-j)}{|x-\bar{x}|^2} + \right. \\
&\qquad \left. + \int_{x_2<0} d^2x \frac{U_C^-(j)U_C^-(1-j)}{|x-\bar{x}|^2} \right) \frac{\tilde{q}^{us^2}}{\eta(\tilde{q})^3} \\
&= \int d^2x \frac{1}{|x-\bar{x}|^2} \int_0^\infty ds \frac{\cosh(\pi us) 2\pi us}{\sinh(2\pi us)} \frac{\tilde{q}^{us^2}}{\eta(\tilde{q})^3} \,. \qquad (5.3)
\end{aligned}$$

By using the modular transformation (3.19), we obtain

$${}_C\langle C|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|C\rangle_C = \int d^2x \frac{1}{|x-\bar{x}|^2} \int_0^\infty ds' \frac{\sqrt{2u}\pi s'}{\tanh(2\pi s')} \frac{q^{us'^2}}{\eta(q)^3} \,, \qquad (5.4)$$

and the spectral density can be read as

$$\rho(s) \propto \frac{s}{\tanh(2\pi s)} \,. \qquad (5.5)$$

This quantity may be derived directly but it seems difficult since it depends on the regularizations. Thus we will only compare below the spectral density of open strings which can be easily compared with the spectral density previously obtained (3.21).

The spectrum of open strings in the presence of orientifold can be read from the Möbius strip amplitudes. It is convenient to use the following characters [19] in the calculation as

$$\hat{\chi}_j(q) = e^{-\pi i(\Delta_j - \frac{c}{24})} \chi_j(-\sqrt{q}) \,, \qquad (5.6)$$

where we are using the characters

$$\chi_s(q) = \frac{q^{us^2}}{\eta(q)^3} \,. \qquad (5.7)$$

The Möbius strip amplitudes are obtained as the overlaps between boundary states and crosscap state as

$$\begin{aligned}
{}_C\langle \Theta|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|C\rangle_C &= \int_{\frac{1}{2}+i\mathbb{R}_+} dj \left(\int_{x_2>0} d^2x \frac{U_\Theta^+(j)U_C^+(1-j)}{|x-\bar{x}|^2} + \right. \\
&\qquad \left. + \int_{x_2<0} d^2x \frac{U_\Theta^-(j)U_C^-(1-j)}{|x-\bar{x}|^2} \right) \hat{\chi}_s(\tilde{q}) \\
&= \int d^2x \frac{1}{|x-\bar{x}|^2} \int_0^\infty ds \frac{\cosh(\pi us)\cos(2\Theta s) 2\pi us}{\sinh(2\pi us)} \hat{\chi}_s(\tilde{q}) \,. \qquad (5.8)
\end{aligned}$$

In the case of Möbius strip amplitudes, the modular transformation can be given by so-called P transformation ($P = T^{1/2}ST^2ST^{1/2}$) [19]. It transforms $\tau \to -1/(4\tau)$ and in this case

$$\frac{se^{2\pi\tau us^2}}{\eta(\tau)^3} = 2\sqrt{u} \int_0^\infty ds' \sin(2\pi uss') \frac{s' e^{2\pi(-1/4\tau)us'^2}}{\eta\left(-\frac{1}{4\tau}\right)^3} \,. \qquad (5.9)$$

Using this modular transformation, we obtain

$$ {}_C\langle\Theta|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|C\rangle_C = \int d^2x \frac{1}{|x-\bar{x}|^2}\int_0^\infty ds' \frac{(\sqrt{u}\pi/4)\sinh(2\pi s')s'}{\cosh(2\pi s')+\cosh\left(\frac{2}{u}\Theta\right)}\hat{\chi}_{s'}(q)\,, \qquad (5.10) $$

and the spectral density can be read as

$$ \rho(s) \propto \frac{\sinh(2\pi s)s}{\cosh(2\pi s)+\cosh\left(\frac{2}{u}\Theta\right)}\,. \qquad (5.11) $$

This density is the same as that of open strings stretched between the branes and their mirrors, namely, the density (3.21) with $\Theta_1 = \Theta$ and $\Theta_2 = -\Theta$.

This correspondence is not accidental. The Möbius strip amplitudes are reconstructed by the information of the annulus amplitudes and the behavior of the open string states under the orientifold operation [24]. As we said in section 2, the orientifold operation can be given by the combination of worldsheet parity reversal Ω and space $\mathbb{Z}_2$ isometries h. In the coordinates (2.2), h acts as

$$ h: \quad \phi\to\phi\,, \quad \gamma\to\bar{\gamma}\,, \quad \bar{\gamma}\to\gamma\,, \qquad (5.12) $$

and for the boundary coordinates as $x \to \bar{x}$ and $\bar{x} \to x$. Therefore the functions (2.4) do not change under this operation and hence the orientifold operation are expected to be independent of j. Moreover the currents are transformed by the orientifold operation as $J^a \to \bar{J}^a$. By following the analysis of [23] and using the above information, we can show that the Möbius strip amplitudes (5.10) can be correctly reconstructed by using the cylinder amplitudes between the boundary states for the mirror branes (3.18). This is an attractive result and from this reason we can rely on our choice of the solutions (5.1).

6. Conclusion

We construct the crosscap states for the orientifolds of euclidean AdS_3 from the solutions of one point functions on $\mathbb{RP}^2$ (5.1). In the classical limit, we can show that these crosscap states describes the orientifolds with correct geometry. The Klein bottle and Möbius strip amplitudes are calculated and the open string spectrum in the system with orientifold is compared to the spectrum of open strings between D-branes and their mirrors.

We have to do more checks to obtain more evidences that our choice of solutions is correct. One way is to make more constraints by using other primaries. This seems to be very complicated but in principle we can do. The other way is to compute the spectral density directly by other methods and to compare with ours. Since our orientifolds have infinite volumes, we have to use some regularizations. In [13], the open string spectrum was derived directly and compared by using the cut-off regularization. The leading terms are removed by using the boundary states with reference boundary conditions Θ_*. Therefore in order to follow their methods in our case, we might have to find the similar reference crosscap states. In our paper, we closely follow the discussions in [12] and the spectral density is identified as the part which scales as the volume of D-branes or orientifolds.

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However, the usual regularization might be given by the cut-off regularization, thus it is important to study the relation between these regularizations.[8]

Compared to the boundary states, many crosscap states are left to be constructed. For example, it would be interesting to construct the crosscap states in Liouville theory or the orientifolds in SU(N) WZW models wrapping on the twisted conjugacy classes like ours. It seems also important to apply to the AdS/CFT correspondence in the system with orientifold.

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A. Several useful formulae

The hypergeometric functions have the following properties under the reparametrizations

$$F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z)\,, \tag{A.1}$$

$$\begin{aligned} F(a,b,c;1-z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,1-c+a+b;z) + \\ &\quad + z^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,1+c-a-b;z)\,. \end{aligned} \tag{A.2}$$

The Gauss recursion formulae for the parameters (a,b,c) are given by

$$cF(a,b,c;z) - (c-b)F(a,b,c+1;z) - bF(a,b+1,c+1;z) = 0\,, \tag{A.3}$$

$$cF(a,b,c;z) + (b-c)F(a+1,b,c+1;z) - b(1-z)F(a+1,b+1,c+1;z) = 0\,, \tag{A.4}$$

$$cF(a,b,c;z) - cF(a+1,b,c;z) + bzF(a+1,b+1,c+1;z) = 0\,. \tag{A.5}$$

We often use the following formulae for Gamma function as

$$\Gamma(1+z) = z\Gamma(z)\,, \tag{A.6}$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}\,, \tag{A.7}$$

$$\Gamma(1+ix)\Gamma(1-ix) = \frac{\pi x}{\sinh(\pi x)}\,, \tag{A.8}$$

where z is an arbitrary complex number and x is a real number.

The Dedekind η function is defined by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{A.9}$$

where $q = \exp(2\pi i\tau)$ and its modular transformation is given by

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)\,, \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)\,. \tag{A.10}$$

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